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Numerical solution of variational problems using exponential sextic spline

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Abstract: In this paper, exponential sextic spline function is used for finding the numerical solution of boundary value problems which arise from the problems of calculus of variations. Spline relations are derived and direct methods of order two, four and six have been obtained. Convergence analysis is briefly discussed. Numerical examples are given to demonstrate the efficiency of the presented method. Comparisons are made to confirm the reliability and accuracy of the proposed technique.

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1 Introduction

The calculus of variations and its extensions are devoted to finding the optimum function that gives the best value of the economic model and satisfies the constraints of a system. A considerable renewal of interest in the classical problems of the calculus of variations has been both from the point of view of mathematics and its applications in physics, engineering and applied mathematics. Several authors obtained analytical and numerical methods for approximating the solution of problems in the calculus of variations such as direct method of Ritz and Galerkin given in [8, 9], orthogonal polynomials [2, 7], Walsh series method [3], Adomain decomposition method [4], Bernstein direct method [5], Haar wavelets [6], Legendre wavelets method [16], Chebyshev finite difference method [17], B-spline collocation method [18], parametric spline method [19, 20] and references therein.

Khan and Aziz [11] used parametric spline approach for the solution of system of second order boundary value problems while Khan [12] presented a numerical method for two point boundary value problems using parametric cubic spline. Recently, Khan and Khandelwal [14] developed a class of methods for solving second order singularly perturbed boundary value problems using nonpolynomial sextic splines. Zarebnia et al. [20] used parametric quintic spline method while Mohammadi et al. [15] used exponential cubic spline to develop a numerical method for approximating the solution of calculus of variation problems. We have used exponential sextic spline for its solution.

The aim of this paper is to construct a new spline method based on exponential spline function of the form $ae^{kx}+be^{-kx}+p_{n-2}(x)$, with $p_{n-2}(x) = \sum_{i=0}^{n-2} m_i x^i$ is an ordinary polynomial of degree $n-2$ and an exponential part to develop numerical methods for obtaining the smooth approximations for finding the extremum of a functional over the specified domain. The exponential sextic spline function proposed in this paper has the form:

$$T_6 = span\left\{1,\ x,\ x^2,\ x^3,\ x^4,\ e^{kx},\ e^{-kx}\right\},$$

where k is a free parameter which can be real or pure imaginary. It will be used to raise the accuracy of the method. The advantage of this method is higher accuracy with the same computational effort.

The paper is organised into six sections. In section 2, we introduce the general form of problems in calculus of variations and explain their relations with boundary value problems. In section 3, we give a brief derivation of exponential sextic spline and some useful relations for solving boundary value problems which arise from problems of calculus of variations. In section 4, exponential sextic spline method approximating the analytical solution of boundary value problem is determined. Convergence analysis is briefly discussed in section 5. Finally, in section 6, numerical examples and comparison with the existing methods are given which demonstrate the practical applicability and superiority of the proposed method.

2 Statement of the problem

The general form of a variational problem is finding extremum of the functional of the form:

$$J[u_1(t),\ u_2(t), \cdots,\ u_n(t)] = \int_a^b F(t,\ u_1(t),\ u_2(t), \cdots,\ u_n(t), u_1'(t),\ u_2'(t), \cdots,\ u_n'(t))dt \tag{2.1}$$

with the given boundary conditions:

$$u_i(a) = A_i,\ u_i(b) = B_i, \quad i = 1, 2, \cdots, n. \tag{2.2}$$

The necessary condition for $u_i(t),\ i=1,2,\cdots,n$, to extremize $J[u_1(t),\ u_2(t),\cdots,\ u_n(t)]$ is to satisfy the Euler-Lagrange equation that is obtained by applying the well known procedure in calculus of variation,

$$\frac{\partial F}{\partial u_i}-\frac{d}{dt}(\frac{\partial F}{\partial u_i'})=0,\ i=1,2\cdots,n, \tag{2.3}$$

subject to the boundary conditions given by (2.2).
In this paper, we consider the special form of variational problem (2.1) as

$$J[u(t)]=\int_a^b F(t,\ u(t),\ u'(t))dt, \tag{2.4}$$

subject to the boundary conditions:

$$u(a)=A_1,\ u(b)=B_1. \tag{2.5}$$

The necessary condition for $u(t)$, to extremize $J[u(t)]$ is that it should satisfy the Euler-Lagrange equation:

$$\frac{\partial F}{\partial u}-\frac{d}{dt}(\frac{\partial F}{\partial u'})=0, \tag{2.6}$$

subject to the boundary conditions given by (2.5).

3 Exponential sextic spline

In order to develop the spline approximation to the solution of problems arises in calculus of variation, the interval $[a,\ b]$ is divided into n equal subintervals using the grid $x_i=a+ih,\ i=0,1,\cdots,n$, where $h=\frac{(b-a)}{n}$. For each segment $[x_i,\ x_{i+1}],\ i=0,1,2\cdots n-1$, we consider the exponential sextic spline $E_i(x)$ of the form:

$$\begin{aligned} E_i(x) &= a_ie^{k(x-x_i)}+b_ie^{-k(x-x_i)}+c_i(x-x_i)^4+d_i(x-x_i)^3+e_i(x-x_i)^2 \\ &\quad +f_i(x-x_i)+g_i, \qquad i=0,1,\cdots,n \end{aligned} \tag{3.1}$$

where $a_i,b_i,c_i,d_i,e_i,f_i,g_i$ are real finite constants and k is a free parameter which will be used to raise the accuracy of the method. If $k\to 0$, then $E_i(x)$ reduces to sextic polynomial spline.

Let $u(x)$ be the exact solution and u_i be an approximation to $u(x_i)$, obtained by the segment $E_i(x)$ of the mixed splines function passing through the points (x_i, u_i) and (x_{i+1}, u_{i+1}). To determine the coefficients of equation (3.1) in terms of $u_i,\ u_{i+1},\ M_i,\ M_{i+1},\ F_i$ and F_{i+1}, we first define:

$$\left.\begin{array}{c} E_i(x_i) = u_i, \quad E_i(x_{i+1}) = u_{i+1}, \\ E_i'(x_i) = m_i, \quad E_i''(x_i) = M_i, \quad E_i''(x_{i+1}) = M_{i+1}, \\ E_i^{(4)}(x_i) = F_i, \quad E_i^{(4)}(x_{i+1}) = F_{i+1}. \end{array}\right\} \tag{3.2}$$

We obtain via a long but straightforward calculation

$$\begin{aligned}
a_i &= \frac{a\{-12(u_{i+1}-u_i)+h^2(4M_i+2M_{i+1})+12m_ih\}+h^4(b)F_{i+1}+h^4(-b-a/2)F_i}{\alpha}, \\
b_i &= \frac{c\{12(u_{i+1}-u_i)-h^2(4M_i+2M_{i+1})-12m_ih\}+h^4(d)F_{i+1}+h^4(c/2-d)F_i}{\alpha}, \\
c_i &= \frac{F_i-a_ik^4-b_ik^4}{24}, \\
d_i &= \frac{u(u_i-u_{i+1})+h(u)m_i+h^2(vM_i+wM_{i+1})+h^4(mF_i+nF_{i+1})}{h^3\alpha}, \\
e_i &= \frac{M_i-a_ik^2-b_ik^2}{2}, \\
f_i &= m_i-a_ik+b_ik, \\
g_i &= u_i-a_i-b_i, \quad \theta = kh \quad \textit{and} \quad i = 0(1)n-1,
\end{aligned}$$

where

$$\begin{aligned}
a &= \theta^4e^{-\theta}-\theta^4, \\
b &= \frac{\theta^4}{2}-4\theta^2-2\theta^2e^{-\theta}+12\theta-12e^{-\theta}-12, \\
c &= \theta^4e^{\theta}-\theta^4, \\
d &= \frac{-\theta^4}{2}+4\theta^2+2\theta^2e^{\theta}+12\theta-12e^{\theta}+12, \\
u &= -\theta^8(e^{\theta}-e^{-\theta}), \\
v &= \frac{-5\theta^8}{12}(e^{\theta}-e^{-\theta})+\theta^6(e^{\theta}-e^{-\theta})-2\theta^5(e^{\theta}+e^{-\theta}-2), \\
w &= \frac{-\theta^8}{12}(e^{\theta}-e^{-\theta})-\theta^6(e^{\theta}-e^{-\theta})+2\theta^5(e^{\theta}+e^{-\theta}-2), \\
m &= \frac{5\theta^6}{12}(e^{\theta}-e^{-\theta})-\theta^5(e^{\theta}+e^{-\theta})+2\theta^3(e^{\theta}+e^{-\theta}-2), \\
n &= \frac{\theta^6}{12}(e^{\theta}-e^{-\theta})+2\theta^5-2\theta^3(e^{\theta}+e^{-\theta}-2), \\
\alpha &= \frac{-\theta^8}{2}(e^{\theta}-e^{-\theta})-6\theta^6(e^{\theta}-e^{-\theta})+2\theta^5(e^{\theta}+e^{-\theta}-2).
\end{aligned}$$

From the continuity of the first, third and fifth derivatives at the point $x = x_i$, we derive the relations for $i = 1, 2, \cdots, n-1$:

$$m_i + m_{i+1} = \frac{2(u_i - u_{i+1})}{h} + \frac{h(M_i - M_{i-1})}{6} - \frac{h^3 p(F_i - F_{i-1})}{\alpha}, \tag{3.3}$$

$$m_i + m_{i+1} = \frac{(u_{i+1} - u_{i-1})}{h} - \frac{h[(\beta/6+\alpha)M_{i+1} + (\beta/2 - 2\alpha)M_i + (\beta/3+\alpha)M_{i-1}]}{\beta} + \frac{h^3(p_1F_{i+1} + q_1F_i + r_1F_{i-1})}{\beta}, \tag{3.4}$$

$$m_i + m_{i+1} = \frac{(u_{i+1} - u_{i-1})}{h} - \frac{h(M_{i+1} + 3M_i + 2M_{i-1})}{6} + \frac{h^3(p_2F_{i+1} + q_2F_i + r_2F_{i-1})}{\gamma}, \tag{3.5}$$

where

$$\begin{aligned}
p &= \frac{\theta^6}{12}(e^\theta - e^{-\theta}) - \frac{\theta^5}{2}(e^\theta + e^{-\theta} + 2) + 8\theta^3(e^\theta + e^{-\theta} + 1) - 2\theta^2(e^\theta - e^{-\theta}) + 24\theta(e^\theta + e^{-\theta} - 2)\\
p_1 &= -\theta^7 - \frac{\theta^6}{2}(e^\theta - e^{-\theta}) + 2\theta^5(e^\theta + e^{-\theta} - 2),\\
q_1 &= \theta^7(e^\theta + e^{-\theta}) - 4\theta^6(e^\theta - e^{-\theta}) - 24\theta^5 + 24\theta^4(e^\theta - e^{-\theta}) - 24\theta^3(e^\theta + e^{-\theta} - 2),\\
r_1 &= -\theta^7 - \frac{3\theta^6}{2}(e^\theta - e^{-\theta}) + 10\theta^5(e^\theta - e^{-\theta}) + 4\theta^5 - 24\theta^4(e^\theta - e^{-\theta}) + 24\theta^3(e^\theta + e^{-\theta} - 2),\\
p_2 &= -\theta^4 + 2\theta^2(e^\theta + e^{-\theta}) + 8\theta^2 - 12(e^\theta + e^{-\theta} - 2),\\
q_2 &= \theta^4(e^\theta + e^{-\theta}) - 6\theta^2(e^\theta + e^{-\theta} + 2) + 12\theta(e^\theta - e^{-\theta}),\\
r_2 &= -\theta^4 + 4\theta^2(e^\theta + e^{-\theta} + 1) - 12\theta(e^\theta - e^{-\theta}) + 12(e^\theta + e^{-\theta} - 2),\\
\beta &= 12\theta^7(e^\theta + e^{-\theta} - 2) - 6\theta^8(e^\theta - e^{-\theta}),\\
\gamma &= 12\theta^4(e^\theta + e^{-\theta} - 2).
\end{aligned}$$

From equations (3.3, 3.4) and (3.5) we obtain on equating the right side of the equality sign

$$u_{i+1} - 2u_i + u_{i-1} = \frac{h^2[(\beta+6\alpha)M_{i+1} + (4\beta - 12\alpha)M_i + (\beta+6\alpha)M_{i-1}]}{6\beta} - \frac{h^4[(p_1\alpha)F_{i+1} + (q_1\alpha + p\beta)F_i + (r_1\alpha - p\beta)F_{i-1}]}{\beta\alpha}, \tag{3.6}$$

$$u_{i+1} - 2u_i + u_{i-1} = \frac{h^2[M_{i+1} + 4M_i + M_{i-1}]}{6} - \frac{h^4[(p_2\alpha)F_{i+1} + (q_2\alpha + p\gamma)F_i + (r_2\alpha - p\gamma)F_{i-1}]}{\alpha\gamma}. \tag{3.7}$$

From above equations we deduce

$$h^4\left\{\frac{p_2}{\gamma}\left(\frac{q_1}{\beta}+\frac{p}{\alpha}\right)-\frac{p_1}{\beta}\left(\frac{q_2}{\gamma}+\frac{p}{\alpha}\right)\right\}F_i = (u_{i+1}-2u_i+u_{i-1})\left\{\frac{p_1}{\beta}-\frac{p_2}{\gamma}\right\} +h^2\left\{\left(\frac{1}{6}+\frac{\alpha}{\beta}\right)\frac{p_2}{\gamma}-\frac{p_1}{6\beta}\right\}(M_{i+1}+M_{i-1}) +h^2\left\{\left(\frac{2}{3}-\frac{2\alpha}{\beta}\right)\frac{p_2}{\gamma}-\frac{2p_1}{3\beta}\right\}M_i. \quad (3.8)$$

(3.8)

Substituting for $F_j(j=i,i\pm 1)$ from equation (3.8) into (3.7), we arrive at the following useful relation in terms of second derivative of spline M_i and u_i:

$$\lambda_1 u_{i-2}+\mu_1 u_{i-1}+\nu_1 u_i+\mu_1 u_{i+1}+\lambda_1 u_{i+2} = h^2(\lambda\, M_{i-2}+\mu\, M_{i-1}+\nu\, M_i+\mu\, M_{i+1}+\lambda\, M_{i+2}), \quad i=2,3,\cdots,n-2, \quad (3.9)$$

where

$$\lambda_1=\tfrac{R}{Z},\quad \mu_1=\tfrac{S}{Z},\quad \nu_1=\tfrac{T}{Z},\quad \lambda=\tfrac{U}{Z},\quad \mu=\tfrac{1}{6}-\tfrac{V}{Z},\quad \nu=\tfrac{2}{3}-\tfrac{W}{Z}$$

and R, S, T, U, V, W and Z are given by

$$R = \frac{p_2}{\gamma}\left(\frac{p_1}{\beta}-\frac{p_2}{\gamma}\right),$$

$$S = 1+\left(\frac{p_1}{\beta}-\frac{p_2}{\gamma}\right)\left(\frac{q_2}{\gamma}+\frac{p}{\alpha}-\frac{2p_2}{\gamma}\right),$$

$$T = -2+\left(\frac{p_1}{\beta}-\frac{p_2}{\gamma}\right)\left(\frac{p_2}{\gamma}-\frac{2q_2}{\gamma}-\frac{3p}{\alpha}+\frac{r_2}{\gamma}\right),$$

$$U = \frac{p_2}{\gamma}\left(\frac{p_1}{6\beta}-\frac{p_2}{\gamma}\frac{\alpha}{\beta}-\frac{p_2}{6\gamma}\right),$$

$$V = \left(\frac{q_2}{\gamma}+\frac{p}{\alpha}\right)\left(\frac{p_2}{6\gamma}+\frac{p_2}{\gamma}\frac{\alpha}{\beta}-\frac{p_1}{6\beta}\right)+\left(\frac{r_2}{\gamma}-\frac{p}{\alpha}\right)\left(\frac{2p_2}{3\gamma}-\frac{2p_2}{\gamma}\frac{\alpha}{\beta}-\frac{2p_1}{3\beta}\right),$$

$$W = \left(\frac{p_2}{\gamma}+\frac{p}{\alpha}-\frac{r_2}{\gamma}\right)\left(\frac{p_2}{6\gamma}+\frac{p_2}{\gamma}\frac{\alpha}{\beta}-\frac{p_1}{6\beta}\right)+\left(\frac{q_2}{\gamma}+\frac{p}{\alpha}\right)\left(\frac{2p_2}{3\gamma}-\frac{2p_2}{\gamma}\frac{\alpha}{\beta}-\frac{2p_1}{3\beta}\right),$$

$$Z = \frac{p_2}{\gamma}\left(\frac{q_1}{\beta}+\frac{p}{\alpha}\right)-\frac{p_1}{\beta}\left(\frac{q_2}{\gamma}+\frac{p}{\alpha}\right).$$

If $\theta\to 0$, then $(\frac{p}{\alpha},\ \frac{p_1}{\alpha},\ \frac{p_1}{\beta},\ \frac{p_2}{\gamma},\ \frac{q_1}{\beta},\ \frac{q_2}{\gamma},\ \frac{r_2}{\gamma},\ \frac{\alpha}{\beta})\to(\frac{1}{360},\ -\frac{1}{36},\ \frac{1}{360},\ \frac{4}{360},\ -$

$\frac{1}{40},\ \frac{7}{120},\ \frac{1}{72},\ -\frac{1}{10})$
and therefore
$(\lambda_1,\ \mu_1,\ \nu_1,\ \lambda,\ \mu,\ \nu) \to (\frac{12}{54},\ \frac{96}{54},\ -\frac{216}{54},\ \frac{4}{540},\ \frac{224}{540},\ \frac{984}{540})$, then spline defined by (3.9) reduces to a sextic spline and the above spline relations reduce to the corresponding ordinary sextic spline relations [13]. To ensure the cost effectiveness, better accuracy and simple applicability of the new method, an alternative way is presented to determine the parameters $\lambda_1,\ \mu_1,\ \nu_1,\ \lambda,\ \mu$ and ν in eq. (3.9), which are actually functions of parameter k. The main advantage of this approach is that it gives a family of second, fourth and sixth order methods with the same computational cost as polynomial spline. The relation (3.9) gives $(n-3)$ linear algebraic equations in $(n-1)$ unknowns $u_i,\ i = 1, 2, \cdots, n-1$. We require two more equations, one at each end of the range of integration. These two equations are given by

$$(i) \qquad \sum_{k=0}^{3} b_k u_k + chu'_0 + h^4 \sum_{k=0}^{5} d_k u_k^{(4)} + t_1 = 0, \qquad i = 1.$$

$$(ii) \qquad \sum_{k=n-3}^{n} b_k u_k - chu'_n + h^4 \sum_{k=n-5}^{n} d_k u_k^{(4)} + t_{n-1} = 0, \qquad i = n-1, \tag{3.10}$$

where b_k , c and d_k are arbitrary parameters to be determined.

3.1 Second-order boundary equations

In order to obtain the boundary equations of second-order, we find that

$(a_0, a_1, a_2, a_3, d_0, d_1, d_2, d_3, d_4, d_5) = (10, -19, 8, 1, 1, -12, 0, 0, 0, 0)$,

$(a_{n-3}, a_{n-2}, a_{n-1}, a_n, d_{n-5}, d_{n-4}, d_{n-3}, d_{n-2}, d_{n-1}, d_n) = (1, 8, -19, 10, 0, 0, 0, 0, -12, 1)$,
and the local truncation error is

$$t_i = \frac{63}{8} h^4 u_i^{(4)} + O(h^5), \qquad i = 1,\ n-1. \tag{3.11}$$

3.2 Fourth-order boundary equations

In order to obtain the boundary equations of fourth-order, we find that

$(a_0, a_1, a_2, a_3, d_0, d_1, d_2, d_3, d_4, d_5) = (10, -19, 8, 1, -\frac{5}{6}, -\frac{101}{12}, -\frac{5}{3}, -\frac{1}{12}, 0, 0)$,

$$(a_{n-3}, a_{n-2}, a_{n-1}, a_n, d_{n-5}, d_{n-4}, d_{n-3}, d_{n-2}, d_{n-1}, d_n) = (1, 8, -19, 10, 0, 0, -\tfrac{1}{12}, -\tfrac{5}{3}, -\tfrac{101}{12}, -\tfrac{5}{6}),$$

and the local truncation error is

$$t_i = -\frac{837}{80} h^6 u_i^{(6)} + O(h^7), \qquad i = 1,\ n-1. \tag{3.12}$$

3.3 Sixth-order boundary equations

In order to obtain the boundary equations of sixth-order, we find that

$$(a_0, a_1, a_2, a_3, d_0, d_1, d_2, d_3, d_4, d_5) = (10, -19, 8, 1, -\tfrac{179}{240}, -\tfrac{1057}{120}, -\tfrac{39}{40}, -\tfrac{41}{60}, \tfrac{61}{240}, -\tfrac{1}{24}),$$

$$(a_{n-3}, a_{n-2}, a_{n-1}, a_n, d_{n-5}, d_{n-4}, d_{n-3}, d_{n-2}, d_{n-1}, d_n) = (1, 8, -19, 10, -\tfrac{1}{24}, \tfrac{61}{240}, -\tfrac{41}{60}, -\tfrac{39}{40}, -\tfrac{1057}{120}, -\tfrac{179}{240}),$$

and the local truncation error is

$$t_i = -\frac{299}{8299} h^8 u_i^{(8)} + O(h^9), \qquad i = 1,\ n-1. \tag{3.13}$$

To obtain the local truncation error $t_i;\ i = 2, 3, \cdots, n-2$, associated with the scheme (3.9), we first rewrite it in the form:

$$\lambda_1 u_{i-2} + \mu_1 u_{i-1} + \nu_1 u_i + \mu_1 u_{i+1} + \lambda_1 u_{i+2} = h^2(\lambda\, u''_{i-2} + \mu\, u''_{i-1} + \nu\, u''_i + \mu\, u''_{i+1} + \lambda\, u''_{i+2}) + t_i; \quad i = 2, 3, \cdots, n-2. \tag{3.14}$$

Using the Taylor's series expansion, the terms $u''_{i-2},\ u''_{i-1}$ etc. are expanded around the point x_i and the expression for $t_i,\ i = 2, 3, \cdots, n-2$ is obtained

$$\begin{aligned} t_i = {} & [2\lambda_1 + 2\mu_1 + \nu_1] u_i + [(4\lambda_1 + \mu_1) - (2\lambda + 2\mu + \nu)] h^2 u''_i \\ & + \left(\frac{16\lambda_1 + \mu_1}{12} - (4\lambda + \mu)\right) h^4 u_i^{(4)} + \left(\frac{64\lambda_1 + \mu_1}{360} - \frac{16\lambda + \mu}{12}\right) h^6 u_i^{(6)} \\ & + \left(\frac{256\lambda_1 + \mu_1}{20160} - \frac{64\lambda + \mu}{360}\right) h^8 u_i^{(8)} + O(h^{10}). \end{aligned} \tag{3.15}$$

By choosing different values of parameters $\lambda_1,\ \mu_1,\ \nu_1,\ \lambda,\ \mu,\ \nu$, methods of different order are obtained.

(i) If we choose $\lambda_1 = 1,\ \mu_1 = 8,\ \nu_1 = -18$ and for any arbitrary choice of $\lambda,\ \mu$ with $\nu = 12 - (2\lambda + 2\mu)$, second order method is obtained.

(ii) If we choose $\lambda_1 = 1,\ \mu_1 = 8,\ \nu_1 = -18$ and $\lambda = -\frac{4}{3},\ \mu = \frac{22}{3},\ \nu = 0$, fourth order method is obtained with truncation error $t_i = \frac{41}{30}h^6 u_i^{(6)}$.

(iii) If we choose $\lambda_1 = 1,\ \mu_1 = 8,\ \nu_1 = -18$ and $\lambda = \frac{1}{30},\ \mu = \frac{56}{30},\ \nu = \frac{66}{30}$, sixth order method is obtained with truncation error $t_i = \frac{1}{504}h^8 u_i^{(8)}$.

4 Application of the method

For the sake of simplicity, we consider the general form of equation (2.6) as follows:

$$u''(x) + f(x)u(x) = g(x),\ x \in [a,b] \tag{4.1}$$

subject to the boundary conditions:

$$u(a) = A_1,\ u(b) = B_1. \tag{4.2}$$

Let $U = u(x_i),\ \bar{U} = (u_i),\ W = (w_i),\ T = (t_i),\ E = (e_i) = U - \bar{U},\ i = 1, 2, \cdots, n-1$ be an exact column vectors, where $U,\ \bar{U},\ T,\ E$ are exact, approximate, local truncation error, discretization error respectively.

We can write the standard matrix equation for the method developed in the following form:

$$\left.\begin{array}{ll} (i) & MU = W + T, \\ (ii) & M\bar{U} = W, \\ (iii) & ME = T, \end{array}\right\} \tag{4.3}$$

where M is a matrix of order $(n-1)$ with

$$M = (M_0 + h^2 BF) \tag{4.4}$$

and M_0, B, F=diag$(f_i),\ i = 1, 2, \cdots, n-1$ are (n-1)×(n-1) matrices.

The five band matrix M_0 has the form:

$$M_0 = J^2 = \begin{bmatrix} a_1 & a_2 & a_3 & & & & & & \\ 8 & -18 & 8 & 1 & & & & & \\ 1 & 8 & -18 & 8 & 1 & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & 1 & 8 & -18 & 8 & 1 \\ & & & & & 1 & 8 & -18 & 8 \\ & & & & & & a_{n-3} & a_{n-2} & a_{n-1} \end{bmatrix}, \tag{4.5}$$

and the matrix B has the form:

$$B = \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & & & & \\ \mu & \nu & \mu & \lambda & & & & & \\ \lambda & \mu & \nu & \mu & \lambda & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & & & \\ & & & & & \ddots & & & \\ & & & & \lambda & \mu & \nu & \mu & \lambda \\ & & & & & \lambda & \mu & \nu & \mu \\ & & & & -d_{n-5} & -d_{n-4} & -d_{n-3} & -d_{n-2} & -d_{n-1} \end{bmatrix}. \tag{4.6}$$

For the $(n-1)$ column vector W, we have

$$w_i = \begin{cases} -a_0A_1 - h^2d_0(g_0 - f_0A_1) \\ -h^2(d_1g_1 + d_2g_2 + d_3g_3 + d_4g_4 + d_5g_5), & i = 1, \\ h^2[\lambda(g_0 - f_0A_1) + \mu g_1 + \nu g_2 + \mu g_3 + \lambda g_4] - A_1, & i = 2, \\ h^2(\lambda g_{i-2} + \mu g_{i-1} + \nu g_i + \mu g_{i+1} + \lambda g_{i+2}), & 3 \leq i \leq n-3, \\ h^2[\lambda(g_n - f_nB_1) + \lambda g_{n-4} + \mu g_{n-3} + \nu g_{n-2} + \mu g_{n-1}] - B_1, & i = n-2, \\ -a_nB_1 - h^2d_n(g_n - f_nB_1) \\ -h^2(d_{n-5}g_{n-5} + d_{n-4}g_{n-4} + d_{n-3}g_{n-3} + d_{n-2}g_{n-2} + d_{n-1}g_{n-1}), & i = n-1. \end{cases} \tag{4.7}$$

5 Convergence analysis

In this section, we discussed the convergence analysis of the method (3.9) along with (3.10) based on exponential sextic spline. Our main purpose is to derive bounds on E. For this, the following lemma is needed.

Leema 5.1 If H is a matrix of order N and $\| H \| < 1$, then there exist $(I+H)^{-1}$ and

$$\| (I+H)^{-1} \| < \frac{1}{1 - \| H \|}.$$

We can rewrite the error equation (4.3)(iii) in the form:

$$E = M^{-1}T = (M_0 + h^2BF)^{-1}T = (1 + h^2M_0^{-1}BF)^{-1}M_0^{-1}T.$$

we get

$$\| E \|_\infty \leq \frac{\| M_0^{-1} \|_\infty \| T \|_\infty}{1 - h^2 \| M_0^{-1} \|_\infty \| B \|_\infty \| F \|_\infty}, \tag{5.1}$$

provided that

$$h^2 \| M_0^{-1} \|_\infty \| B \|_\infty \| F \|_\infty < 1. \tag{5.2}$$

Now

$$\| B \|_\infty = 1 \quad \text{and} \quad \| F \|_\infty \leq \| f \| = \max_{a \leq x \leq b} |f(x)|.$$

By using Henrici [10] we have

$$\| M_0^{-1} \|_\infty = \frac{(b-a)^2}{8h^2} = O(h^{-2}). \tag{5.3}$$

Now, using equations (3.11)-(3.13), we investigate the convergence analysis of second-order, fourth-order and sixth-order methods.

Case (i) *Second-order method*

For $(\lambda, \mu, \nu) = (0, 0, 12)$, and using equation (3.11), we have

$$\| T \|_\infty = \frac{63}{8} h^4 M_4, \quad M_4 = \max_{a \leq x \leq b} |u^{(4)}(x)|, \tag{5.4}$$

then from (5.1)-(5.4), it follows that

$$\| E \|_\infty \leq \frac{63\xi h^2 M_4}{8(1 - \xi |f(x)|)} = K_2 h^2 = O(h^2), \tag{5.5}$$

where

$$\xi = \frac{(b-a)^2}{8} \quad \text{and} \quad K_2 = \frac{63\xi M_4}{8(1-\xi|f(x)|)},$$

which shows that method developed for the solution of second order boundary value problem is second-order convergent.

Case (ii) *Fourth-order method*

For $(\lambda, \mu, \nu) = \frac{1}{3}(-4, 22, 0)$, and using equation (3.12), we have

$$\| T \|_\infty = \frac{837}{80} h^6 M_6, \quad M_6 = \max_{a \leq x \leq b} |u^{(6)}(x)|, \tag{5.6}$$

then from (5.1)-(5.3) and (5.6), it follows that

$$\| E \|_\infty \leq \frac{837\xi h^4 M_6}{80(1-\xi|f(x)|)} = K_4 h^4 = O(h^4), \tag{5.7}$$

where

$$\xi = \tfrac{(b-a)^2}{8} \quad \text{and} \quad K_4 = \tfrac{837\xi M_6}{80(1-\xi|f(x)|)},$$

which shows that method developed for the solution of second order boundary value problem is fourth-order convergent.

Case (iii) *Sixth-order method*

For $(\lambda, \mu, \nu) = \frac{1}{30}(1, 56, 66)$, and using equation (3.13), we have

$$\| T \|_\infty = \frac{229}{8299} h^8 M_8, \quad M_8 = \max_{a \leq x \leq b} |u^{(8)}(x)|, \tag{5.8}$$

then from (5.1)-(5.3) and (5.8), it follows that

$$\| E \|_\infty \leq \frac{229\xi h^6 M_8}{8299(1-\xi|f(x)|)} = K_6 h^6 = O(h^6), \tag{5.9}$$

where

$$\xi = \tfrac{(b-a)^2}{8} \quad \text{and} \quad K_6 = \tfrac{229\xi M_8}{8299(1-\xi|f(x)|)},$$

which shows that method developed for the solution of second order boundary value problem is sixth-order convergent.

We summarize the above results in the following theorem:

Theorem: Let $u(x)$ be the exact solution of second-order boundary value problem and let $u_i, i = 1, 2, \cdots, n$ be the numerical solution obtained by the difference scheme (ii) in (4.3). Further, if $e_i = u(x_i) - u_i$, then

(1) $\| E \| = O(h^2)$, is a second-order method which is given by Eqn.(5.5),
(2) $\| E \| = O(h^4)$, is a fourth-order method which is given by Eqn.(5.7),
(3) $\| E \| = O(h^6)$, is a sixth-order method which is given by Eqn.(5.9),

neglecting all errors due to rounding off.

6 Numerical examples

The numerical methods outlined in the previous sections were tested on the following problems. All the computations were performed by using MATLAB. We also compare our method with the existing methods which demonstrate efficiency and better accuracy.

Example 1. Consider the following problem for finding the extremals of the functional, which is discussed in [15, 18]:

$$\min J[u(x)] = \int_0^1 (u(x) + u'(x) - 4\exp(3x))^2 dx, \tag{6.1}$$

subjected to the boundary conditions:

$$u(0) = 1, \;\; u(1) = \exp(3). \tag{6.2}$$

The corresponding Euler-Lagrange equation is

$$u''(x) - u(x) = 8\exp(3x), \tag{6.3}$$

with boundary conditions given by (6.2). The exact solution of the above problem is

$$u(x) = \exp(3x). \tag{6.4}$$

Example 2. Consider the following problem for finding the extremals of the functional, which is discussed in [15]:

$$\min J[u(x)] = \int_0^1 ((u'(x))^2 + xu'(x) + (u(x))^2) dx, \tag{6.5}$$

subjected to the boundary conditions:

$$u(0) = 0, \;\; u(1) = \frac{1}{4}. \tag{6.6}$$

The corresponding Euler-Lagrange equation is

$$u''(x) - u(x) = -\frac{1}{2}, \tag{6.7}$$

with boundary conditions given by (6.6). The exact solution of the above problem is

$$u(x) = \frac{1}{2} + \frac{2 - \exp}{4(\exp^2 - 1)}\exp(x) + \frac{\exp(1 - 2\exp)}{4(\exp^2 - 1)}\exp(-x). \tag{6.8}$$

Example 3. Consider the following problem for finding the extremals of the functional, which is discussed in [18]:

$$J[u_1(x),\ u_2(x)] = \int_0^{\pi/2} (u_1'^2(x) + u_2'^2(x) + 2u_1(x)u_2(x))dx, \tag{6.9}$$

subjected to the boundary conditions:

$$\left.\begin{aligned} u_1(0) = 0,\ u_1(\pi/2) = 1, \\ u_2(0) = 0,\ u_2(\pi/2) = -1. \end{aligned}\right\} \tag{6.10}$$

The corresponding Euler-Lagrange equations are

$$\left.\begin{aligned} u_1''(x) - u_2(x) = 0, \\ u_2''(x) - u_1(x) = 0. \end{aligned}\right\} \tag{6.11}$$

with boundary conditions given by (6.10). The exact solutions of the above problem are

$$u_1(x) = \sin(x),\ u_2(x) = -\sin(x). \tag{6.12}$$

Example 4. Consider the following problem for finding the extremals of the functional:

$$\min J[u(x)] = \int_0^{\pi/4} ((u(x))^2 - (u'(x))^2)dx, \tag{6.13}$$

subjected to the boundary conditions:

$$u(0) = 1,\ u(\pi/4) = \sqrt(2). \tag{6.14}$$

The corresponding Euler-Lagrange equation is

$$u''(x) + u(x) = 0, \tag{6.15}$$

with boundary conditions given by (6.14). The exact solution of the above problem is

$$u(x) = \sin(x) + \cos(x). \tag{6.16}$$

The observed maximum absolute errors corresponding to the examples 1-4 for our second, fourth and sixth-order methods are briefly summarized in tables 1-4. Comparison with other existing methods are also listed in tables. These tables show that our class of methods are more accurate than the existing spline methods.

Table 1: Observed maximum absolute errors, Example 1.

Methods $\downarrow$	$\mathbf{n=8}$	$\mathbf{n=16}$	$\mathbf{n=32}$	$\mathbf{n=64}$	$\mathbf{n=128}$
Second order method ($\lambda=0,\ \mu=0,\ \nu=12$)	1.370×10^{-1}	3.426×10^{-2}	8.564×10^{-3}	2.141×10^{-3}	5.353×10^{-4}
Fourth order method ($\lambda=-\frac{4}{3},\ \mu=\frac{22}{3},\ \nu=0$)	1.071×10^{-2}	7.773×10^{-4}	5.065×10^{-5}	3.202×10^{-6}	2.007×10^{-7}
Sixth order method ($\lambda=\frac{1}{30},\ \mu=\frac{56}{30},\ \nu=\frac{66}{30}$)	9.638×10^{-6}	5.250×10^{-8}	4.784×10^{-10}	9.774×10^{-12}	6.910×10^{-13}
Mohammadi et. al [15]	4.7907×10^{-4}	3.0066×10^{-5}	1.8811×10^{-6}	1.1760×10^{-7}	7.3505×10^{-9}
Zarebnia et. al [18]	6.9109×10^{-2}	1.7165×10^{-2}	4.2845×10^{-3}	1.0707×10^{-3}	2.6764×10^{-4}

Table 2: Observed maximum absolute errors, Example 2.

Methods $\downarrow$	$\mathbf{n=8}$	$\mathbf{n=16}$	$\mathbf{n=32}$	$\mathbf{n=64}$	$\mathbf{n=128}$
Second order method ($\lambda=0,\ \mu=0,\ \nu=12$)	1.003×10^{-4}	2.504×10^{-5}	6.281×10^{-6}	1.570×10^{-6}	3.925×10^{-7}
Fourth order method ($\lambda=-\frac{4}{3},\ \mu=\frac{22}{3},\ \nu=0$)	9.3875×10^{-7}	6.4685×10^{-8}	4.157×10^{-9}	2.614×10^{-10}	1.635×10^{-11}
Sixth order method ($\lambda=\frac{1}{30},\ \mu=\frac{56}{30},\ \nu=\frac{66}{30}$)	6.504×10^{-11}	2.434×10^{-13}	6.217×10^{-15}	8.715×10^{-15}	2.148×10^{-14}
Mohammadi et. al [15]	3.9058×10^{-8}	2.4442×10^{-9}	1.5266×10^{-10}	9.5825×10^{-12}	5.9924×10^{-13}

Table 3: Observed maximum absolute errors, Example 3.

Methods ↓	**n = 8**	**n = 16**	**n = 32**	**n = 64**	**n = 128**
$\mathbf{u_1(x)}$					
Second order method ($\lambda=0,\ \mu=0,\ \nu=12$)	1.339×10^{-3}	3.387×10^{-4}	8.458×10^{-5}	2.114×10^{-5}	5.284×10^{-6}
Fourth order method ($\lambda=-\frac{4}{3},\ \mu=\frac{22}{3},\ \nu=0$)	3.139×10^{-5}	2.168×10^{-6}	1.383×10^{-7}	8.686×10^{-9}	5.435×10^{-10}
Sixth order method ($\lambda=\frac{1}{30},\ \mu=\frac{56}{30},\ \nu=\frac{66}{30}$)	5.522×10^{-9}	1.941×10^{-11}	4.629×10^{-13}	9.215×10^{-15}	4.269×10^{-14}
Zarebnia et. al [18]	8.9855×10^{-4}	2.2507×10^{-4}	5.6321×10^{-5}	1.4082×10^{-5}	3.5207×10^{-6}
$\mathbf{u_2(x)}$					
Second order method ($\lambda=0,\ \mu=0,\ \nu=12$)	1.339×10^{-3}	3.387×10^{-4}	8.458×10^{-5}	2.114×10^{-5}	5.284×10^{-6}
Fourth order method ($\lambda=-\frac{4}{3},\ \mu=\frac{22}{3},\ \nu=0$)	3.139×10^{-5}	2.168×10^{-6}	1.383×10^{-7}	8.686×10^{-9}	5.435×10^{-10}
Sixth order method ($\lambda=\frac{1}{30},\ \mu=\frac{56}{30},\ \nu=\frac{66}{30}$)	5.522×10^{-9}	1.941×10^{-11}	4.629×10^{-13}	9.215×10^{-15}	4.269×10^{-14}
Zarebnia et. al [18]	8.9855×10^{-4}	2.2507×10^{-4}	5.6321×10^{-5}	1.4082×10^{-5}	3.5207×10^{-6}

Table 4: Observed maximum absolute errors, Example 4.

Methods ↓	**n = 8**	**n = 16**	**n = 32**	**n = 64**	**n = 128**
Second order method ($\lambda=0,\ \mu=0,\ \nu=12$)	1.719×10^{-4}	4.275×10^{-5}	1.067×10^{-5}	2.669×10^{-6}	6.672×10^{-7}
Fourth order method ($\lambda=-\frac{4}{3},\ \mu=\frac{22}{3},\ \nu=0$)	9.925×10^{-7}	6.824×10^{-8}	4.360×10^{-9}	2.741×10^{-10}	1.735×10^{-11}
Sixth order method ($\lambda=\frac{1}{30},\ \mu=\frac{56}{30},\ \nu=\frac{66}{30}$)	3.494×10^{-11}	1.083×10^{-13}	2.465×10^{-14}	1.705×10^{-13}	5.982×10^{-13}

Concluding remarks

Exponential sextic spline functions are used to develop a class of numerical methods for finding the numerical solution of problems arises in calculus of variation. The computations associated with the examples discussed above were performed by using MATLAB 7. The methods are computationally efficient and can be easily implemented on a computer. The present method enable us to approximate the solution at every point of the range of integration. Comparison of the method is also depicted through tables 1-4 which shown that our methods perform better in the sense of accuracy and applicability.

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